Simple model to estimate the contribution of atmospheric CO_2 to the Earth's greenhouse effect

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We show how the CO_2 contribution to the Earth's greenhouse effect can be estimated from relatively simple physical considerations and readily available spectroscopic data. In particular, we present a calculation of the "climate sensitivity" (that is, the increase in temperature caused by a doubling of the concentration of CO_2) in the absence of feedbacks. Our treatment highlights the important role played by the frequency dependence of the CO_2 absorption spectrum. For pedagogical purposes, we provide two simple models to visualize different ways in which the atmosphere might return infrared radiation back to the Earth. The more physically realistic model, based on the Schwarzschild radiative transfer equations, uses as input an approximate form of the atmosphere's temperature profile, and thus includes implicitly the effect of heat transfer mechanisms other than radiation.
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I. INTRODUCTION

The study of the "greenhouse effect" due to the CO_2 in the Earth's atmosphere has a long and interesting history.[1] Although uncertainty remains regarding the long-term effects on the Earth's temperature of various feedback mechanisms that would be expected to accompany an increase in atmospheric CO_2, the basic physics of the CO_2-induced warming in the absence of such feedbacks is (or should be) uncontroversial. Nonetheless, we feel there is a lack of a simple presentation of this effect at a sufficiently detailed, technical level to make it possible for an interested, but non-specialist physicist, or physics student, to get a quick feel for the numbers involved.

The purpose of this paper is to offer such a presentation. Our goal is to provide a self-contained treatment that is more mathematical than the recent, very good introductory article in *Physics Today* by Pierrehumbert,[2] while still remaining at a much simpler level than the same author's recent textbook[3] (which is intended to prepare students to do original research in climate science).[4] Our model is also somewhat more advanced than the one presented in Sec. 8.5 of the recent textbook by Andrews,[5] which, however, is a very good introduction to many related topics that we will only mention in passing here. We should also mention the relatively recent article in this same journal by Tomizuka,[6] which has a number of points in common with ours; however, its numerical approach does not lend itself well to the derivation of approximate analytical expressions (in particular, for the "climate sensitivity") such as the ones we will present here. Nonetheless, it is a useful reference for the effect of other greenhouse gases not covered here.

Since we are not, ourselves, atmospheric physicists, our chosen approach may be a little unconventional from the point of view of this discipline, but we believe it should feel natural to a physicist approaching the problem "from scratch," with only a basic understanding of thermodynamics and atomic and molecular spectroscopy. Accordingly, we have tried to introduce only a minimum of technical, specialized concepts,[7] and, in the time-honored physics tradition of back-of-the-envelope calculations, we have kept our models as simple as possible. Nonetheless, our final estimates for a couple of important quantities—namely, the radiative forcing equivalent of CO_2, and the "climate sensitivity" to a doubling of the CO_2 concentration—turn out not to be very far from their accepted values. We believe, therefore, that our treatment may help demystify the subject somewhat, and hence be valuable to interested physicists and physics students, some of whom may be confronted with questions about CO_2-induced global warming and would like to have a calculation, of, at least, the foundation of the effect, in a relatively compact form that they can follow and check for themselves.

We begin (in Sec. II) by presenting a brief explanation of the greenhouse effect for an idealized Earth with a uniform surface temperature. Greenhouse warming is then related to the fraction x of the radiation emitted by the surface of the Earth that is prevented from escaping out to space by the "greenhouse gases." The remainder of the paper is devoted to calculating the contribution of atmospheric CO_2 to this fraction. This requires us, first, to consider the absorption spectrum of CO_2, for which we introduce a somewhat crude approximation in Sec. III (based on freely available spectroscopic data), and then, to develop a model for what happens to the absorbed photon's energy in the atmosphere. Two such models are presented: in the first one (Sec. IV), the history of an individual photon is treated as a random walk, consisting of absorption and reemission events by molecules in a hypothetical "static" atmosphere. This model neglects convection as well as any other form of energy redistribution or transfer and hence overestimates the actual greenhouse warming; otherwise put, it provides us with an upper limit for the warming potential of atmospheric CO_2 alone.

Our second model (Sec. V), while still highly simplified, involves the radiative transfer equation in a one-dimensional, temperature-stratified atmosphere, which could be written down immediately after reading Pierrehumbert's *Physics Today* article.[2] This is sometimes referred to as Schwarzschild's equation, since it is closely related to one introduced by Schwarzschild in 1906 to describe the Sun's atmosphere.[8] We derive approximate analytical solutions to this equation, using as input a standard approximation to the temperature profile of the Earth's atmosphere, and, as mentioned above, we find results in fairly good agreement with currently accepted values. We also point out, by a conservation-of-energy argument, that this model implicitly includes non-radiative cooling of the surface of the Earth by the atmosphere.

Finally, in Sec. VI, we compare the analytical results of Sec. V to the result of a numerical integration over absorption lines using the MODTRAN package, for which a Web-based interface is also freely available.[9] Again, the agreement, for the appropriate choice of parameters, is reasonably good. This may have the positive effect of making the MODTRAN calculator appear less like a black box, increasing the user's confidence in its results for other parameter settings, which can be used to explore situations well beyond what we consider here.

II. THE GREENHOUSE EFFECT

The basic physics behind the Earth's greenhouse effect can be understood as follows.[10] The Earth receives, on average, $I_0 = 1361$ W/m^2 of power from the sun, which it absorbs as a disk with area πR_e^2. Assuming that it radiates as a spherical blackbody with a surface $4\pi R_e^2$, its steady state temperature T_0 would be determined by the Stefan-Boltzmann law as

$$\sigma_{SB} T_0^4 \times 4\pi R_e^2 = (1-\alpha) I_0 \times \pi R_e^2, \quad (1)$$

where $\sigma_{SB} = 5.67 \times 10^{-8}$ W/m^2 K^4 is the Stefan-Boltzmann constant, and α is the albedo of the Earth, that is, the fraction of solar intensity that is directly reflected out to space without absorption. With $\alpha = 0.3$, one obtains $T_0 = 255$ K.

Of course, the surface of the Earth is not at a uniform temperature; in addition to the difference between the day and night sides, there are important variations with latitude and topography. Nonetheless, the notion of an average temperature is a useful one, and observations from space are consistent with a blackbody radiation spectrum, only with "gaps" corresponding to absorption by greenhouse gases (see, for instance, Fig. 3 of Pierrehumbert's article[2]). An analysis of some of the issues associated with defining an average surface temperature for a rotating planet with thermal inertia has been presented in Ref. 11.

In any case, observations indicate that the average temperature of the Earth's surface is actually rather higher than the value of T_0 calculated from Eq. (1), closer, in fact, to 288 K. The "greenhouse effect" explanation, in its simplest form, is as follows. The Earth's atmosphere contains a number of gases—of which the most important, at their present concentrations, are water vapor and carbon dioxide—that let through most of the sun's radiation, but absorb strongly some of the infrared radiation emitted by the surface.

Let T be the actual average surface temperature. The rate, or intensity, of radiation (in, say, W/m^2) is then $\sigma_{SB} T^4$. Assume that the net effect of the atmosphere is to reduce the total intensity that eventually makes it out to space by a fraction x of this quantity. The radiative equilibrium equation then becomes, instead of Eq. (1),

$$(1-x)\sigma_{SB} T^4 = \frac{1-\alpha}{4} I_0, \quad (2)$$

from which a new (higher) equilibrium temperature T results. Combining Eqs. (1) and (2), we see that

$$T = \frac{T_0}{(1-x)^{1/4}}. \quad (3)$$

If $T = 288$ and $T_0 = 255$ K, we see that we must have $x \simeq 0.39$.

The reason for writing Eqs. (2) and (3) in terms of fractions, rather than absolute intensities, is that physically one expects the actual power absorbed, or "blocked," to be (at least as a first approximation) proportional to the power emitted, which depends strongly on the temperature; whereas one expects the ratio x of power absorbed to emitted to depend on the temperature only weakly, or indirectly, through other variables such as the concentrations of the greenhouse gases, or the temperature lapse rate. The dependence of such variables on temperature is in fact an example of the "feedbacks" that we shall ignore here.

Several additional remarks may be in order. First, even though many heat transfer processes may be going on inside the atmosphere, and between the surface and the atmosphere, the ultimate mechanism by which the Earth releases energy to space is radiation, and so Eq. (2) does not leave anything out, in principle, as long as x is calculated correctly. Second, it is possible to move the term $x\sigma_{SB} T^4 \equiv \Delta I$ (the "blocked" intensity) in Eq. (2) to the right-hand side, where it takes a positive sign, and hence looks as an effective *increase* in the intensity of incoming solar radiation; it is for this reason that the infrared-radiation blocking effect of greenhouse gases is often described as an equivalent increase in "radiative forcing." Last, note that, in fact, the "blocking" *must* physically involve radiation from the atmosphere to the surface, since (in steady state) the "blocked" radiation must ultimately return to the surface: the whole point being that, as a consequence of the blocking, the surface is incapable of cooling off as fast as a freely radiating blackbody at temperature T naturally would. In dynamic terms, one may think of this "back-radiation" as "initially" leading to an increase in the average temperature of the Earth, and with it its radiation rate, until, as expressed by Eq. (2), an equilibrium temperature T is reached at which the *net* energy loss to space matches the energy input from the sun. Physical models for how the "blocked" radiation might be returned to the surface are presented in Secs. IV and V.

Our concern will be with estimating the fraction of the power that is blocked by atmospheric CO_2, which we may denote by x_{CO_2}. As mentioned above, there are other greenhouse gases in the atmosphere, and therefore, in general, we will have $x = x' + x_{CO_2}$. Let $T \simeq 288$ K be the observed temperature in the presence of all the greenhouse gases, and let T' be the temperature one would have in the absence of CO_2; that is, we have $(1-x)T^4 = (1-x'-x_{CO_2})T^4 = (1-x')T'^4 = T_0^4$. Eliminating $1-x'$ we obtain T' as

$$T' = \frac{T}{\left(1+\left(\frac{T}{T_0}\right)^4 x_{CO_2}\right)^{1/4}} \simeq \frac{288}{(1+1.63x_{CO_2})^{1/4}}\,\mathrm{K}. \quad (4)$$

We can use this equation to estimate the overall contribution of CO_2 to the current temperature of the Earth; that is, the difference $T - T'$. The basic assumption made in doing so is that x' does not change; that is, we are comparing the current situation to a hypothetical one in which there is no CO_2 in the atmosphere, but all the other greenhouse gases are still present in such a concentration that they still block the same *fraction* of the outgoing radiation as they do now. This is clearly unrealistic, for several reasons: first, because the absorption bands of water vapor and CO_2 overlap in part, complete removal of CO_2 would result in *more* absorption by water vapor, at the same concentration; neglecting this overlap is therefore one of the important simplifying assumptions of our model. On the other hand, as Pierrehumbert[2] notes, a colder Earth would hold *less* water vapor in the atmosphere, which would tend to reduce the water vapor greenhouse effect in the absence of CO_2. Neglecting this latter effect, however, is consistent with our stated purpose to ignore "feedbacks" in order to keep the problem manageable.[12]

Always with the above caveat, we can also use Eq. (3) to estimate the change in T arising from a small enough change in the concentration of CO_2, and hence, in x_{CO_2}, all other things being equal

$$\delta T = \frac{1}{4}\frac{T_0}{(1-x)^{5/4}}\delta x_{CO_2} = \frac{1}{4}\frac{T^5}{T_0^4}\delta x_{CO_2} \simeq 117\delta x_{CO_2}\mathrm{K}. \quad (5)$$

Since Eq. (5) involves a first-order expansion about current conditions, the reference radiation rate for the calculation of both x and δx_{CO_2} is the radiation rate $\sigma_{SB}T^4$ at the current temperature; that is, $\delta x_{CO_2} = \delta(\Delta I)/\sigma_{SB}T^4$, where $\delta(\Delta I)$ is the change in the blocked intensity resulting from a change in the CO_2 concentration n. This quantity, the CO_2 "radiative forcing equivalent," is currently estimated[13] as $5.35\ln(n/n_0)$ W/m^2, or 3.71 W/m^2 for a doubling of n. When put together with $\sigma_{SB}T^4 = 390$ W/m^2, the latter figure gives $\delta x_{co_2} = 9.51\times10^{-3}$, and hence, by Eq. (5), $\delta T = 1.1$ K per doubling, an often-quoted figure for the CO_2 "climate sensitivity" in the absence of feedbacks. In Sec. V, we show how the correct order of magnitude for this quantity can be estimated from a simplified treatment of the CO_2 absorption spectrum (developed in Sec. III) and the radiative transfer equations.

III. THE CO_2 ABSORPTION SPECTRUM

As a triatomic molecule, CO_2 has many strong (dipole-allowed) transitions in the infrared, since both the bending and stretching modes of vibration readily generate dipole moments (see, for instance, the discussion in Sec. 4.4.2 of Ref. 3). Of particular interest for the Earth's greenhouse effect are the set of transitions near 667 cm^{-1} (15 μm), which is close to the peak of the Planck radiation curve for a 288 K blackbody. Rotational states and isotopic shifts result in literally thousands of separate lines in the region between 550 and 800 cm^{-1}. Spectral properties for all these lines are available in the HITRAN database;[14] a convenient online interface to this database may be found at SpectralCalc.com.[15] In what follows we indicate how these data can be used to obtain useful approximations to the very complicated absorption spectrum.

In the HITRAN database, each line is assigned an "intensity" S_i [in units of cm^{-1}/(molecule $\times$ cm^{-2})]. At normal atmospheric pressure, all of the lines may be assumed to be pressure broadened; the half-width at half maximum, γ_i, is also listed for each transition. If ν_i is the central frequency for each transition (ignoring pressure shifts), the total absorption cross-section at any given frequency could be expressed as

$$\sigma(\nu) = \sum_i \frac{S_i}{\pi}\frac{\gamma_i}{\gamma_i^2 + (\nu-\nu_i)^2}, \quad (6)$$

where the sum runs over all transitions [see the Appendix to the article on the 1996 Edition of HITRAN (Ref. 16) for details, especially Eqs. (A14) and (A15)].

In practice, however, it is found that the very far wings of these transitions exhibit a decay faster than the pure Lorentzian form assumed in Eq. (6). According to problem 4.17 of Ref. 3, there are various empirical far-tail line shapes in use, but a simple approach is to just truncate the tail at a fixed number of line widths (for reference, a typical value for γ_i is about 0.07 cm^{-1}). Such a truncation, at $|\nu-\nu_i| > 100\gamma_i$, has been used to produce the spectrum shown in the rapidly varying dotted curve in Fig. 1 (the choice of $100\gamma_i$ was suggested in the same reference quoted above). Without it, the center of the band would look very much the same, but at the edges, above $\nu = 750$ cm^{-1} and below $\nu = 600$ cm^{-1}, the spectrum would actually be dominated by the far tail of the central line; that is, it would not fall off quite as fast as shown in the figure. This could make our estimates of the parameters $r_\pm$ below [see Eq. (8)] smaller by 7% or more, depending on how much of the spectrum we chose to include in our fits.

The dotted curve in Fig. 1 has been plotted at a resolution of 0.1 cm^{-1}. Since this kind of resolution is often impractical, many calculations make use of "band models" (see, for instance, Kiehl and Ramanathan[17]) where an effective band strength is defined over suitable coarse-graining intervals. The dashed line in Fig. 1 shows the result of a simple average of the spectrum over frequency intervals of width $\Delta\nu = 5$ cm^{-1} (for reference, each such interval may contain of the order of 500 partly overlapping lines). We have chosen $\Delta\nu = 5$ cm^{-1} because this is also what is used in Fig. 1 of the paper by Kiehl and Ramanathan,[17] and our result has obvious similarities with that figure, but we emphasize that our simple average is definitely *not* the way the "professionals" compute an effective band strength for practical calculations; indeed, many factors that we will ignore here need to be taken into consideration for such calculations—most importantly the dependence of the spectrum on pressure (for an introduction to all these complications, see Sec. 4.4 of Ref. 3).

Note that if Eq. (6) is averaged over a frequency interval of width $\Delta\nu \gg \gamma_i$ around some central frequency, and the effect of the Lorentzian tails outside of this interval is neglected, one could, with little error, just replace the integral of all the Lorentzian functions by π, and restrict the sum

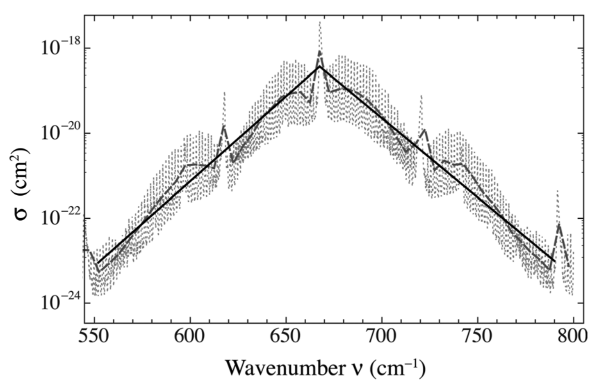

Fig. 1. Absorption cross-section, in cm^2, for a CO_2 molecule as a function of frequency around 15 μm wavelength (light gray dotted curve); note the logarithmic scale. Also shown are a "coarse grained" spectrum (medium gray dashed curve) obtained by averaging over intervals of width 5 cm^{-1}, and a drastically simplified version (black, solid line) that we use for the analytical order-of-magnitude estimates.

to only the lines centered within the interval; if these are more or less uniformly distributed, then only a small fraction (of the order of $\gamma_i/\Delta\nu$) of them is close enough to the edges to introduce an appreciable error. Hence, a much simpler form for the "coarse-grained" spectrum shown in Fig. 1 is just

$$\sigma(\nu_k) = \frac{1}{\Delta\nu} \sum_{|\nu_i - \nu_k| \le \Delta\nu/2} S_i. \tag{7}$$

That is to say, simply assign to each "coarse-grained interval" the average strength of all the lines that it contains. For our choice of $\Delta\nu = 5\ \mathrm{cm}^{-1}$, the result of this prescription is virtually indistinguishable from the dashed line in Fig. 1.

Since the main goal of this paper is to derive relatively simple analytical approximations that make it easy to grasp the basic physics, rather than strive for high numerical accuracy, we will, for all the calculations that follow, simplify the absorption spectrum even further, by replacing it with the triangular (in a logarithmic plot) function shown as a solid line in Fig. 1. This is the result of a least-squares fit to the coarse-grained data, in the interval $550\,\mathrm{cm}^{-1} < \nu < 790\ \mathrm{cm}^{-1}$, and is given by

$$\sigma(\nu) = \sigma_0 \exp[-r_\pm|\nu - \nu_0|] \tag{8}$$

with $\nu_0 = 667.5\ \mathrm{cm}^{-1}$, and different slopes r_+ and r_- depending on whether $\nu > \nu_0$ or $\nu < \nu_0$, respectively. The parameters from the fit are $\sigma_0 = 3.71 \times 10^{-23}\ \mathrm{m}^2$, $r_- = 0.092$ cm, and $r_+ = 0.086$ cm. The importance of the slopes $r_\pm$, which characterize the approximately exponential decrease in the absorption power of the CO_2 molecules away from resonance, will become apparent in the next couple of sections. As for σ_0, we can take it as an estimate of the typical absorption cross-section seen by an infrared photon of frequency close to the center of the absorption band, say between 650 and $690\,\mathrm{cm}^{-1}$, and start to explore its implications. Suppose this photon has been emitted upwards at the surface of the Earth. How high in the atmosphere will it travel before it is absorbed?

Consider a column of air of height l and cross-sectional area A. If n_0 is the number density of CO_2 molecules near the surface of the Earth, the number of molecules in the column is n_0Al. If each molecule appears to the photon as an absorbing disk of area σ_0, and they are randomly spread horizontally, then on average it takes A/σ_0 molecules of CO_2 to block the photon's upward path completely (the other gases in the atmosphere are virtually transparent at these frequencies). Setting these two numbers equal, we get the photon's "mean free path"[18] $l = 1/n_0\sigma_0$. At the current CO_2 concentration of about 390 ppm (parts per million), we have near the surface of the Earth ($T \sim 288$ K, pressure ~ 1 atm) $n_0 \simeq 9.91 \times 10^{21}$ molecules/m^3. Putting this together with $\sigma_0 = 3.7 \times 10^{-23}\ \mathrm{m}^2$, we get $l = 2.7$ m. Hence, the photon does not get very far at all. CO_2 may be transparent to visible light, and its concentration measured in parts per million, but as far as a 15 μm photon traveling upwards is concerned, the bottom of the atmosphere might as well be an impenetrable wall of CO_2.

Clearly, the situation is different for radiation at frequencies towards the edges of the band (say, below 580 or above 760 cm^{-1}), where σ has fallen off by several orders of magnitude; especially since, once the photon makes it to a substantial height, the density of CO_2 goes down as well. It is radiation at these frequencies that is most sensitive to changes in the atmospheric CO_2 concentration. We may get an idea of what happens from the following simplified model, which does *not* really describe the atmosphere of the Earth, but allows us to put an upper bound to the greenhouse potential of CO_2 alone.

IV. FIRST MODEL: A STATIC ATMOSPHERE WITHOUT MOLECULAR COLLISIONS

Suppose that each photon emitted upwards by the Earth performs the following one-dimensional "random walk." After traveling a distance $l = 1/\sigma n$ (where n is the local CO_2 density) it is absorbed, after which it is reemitted, with a 50–50 probability of being sent upwards in the atmosphere or downwards, back to the Earth. If it goes upwards, then we assume that after traveling another distance l it is absorbed again, and so on. The question is, what is the probability that it will eventually get back to the Earth, or, alternatively, eventually escape into space?

This is actually a well-known problem in probability theory, known as a "classical ruin problem."[19] If we start keeping track of the photon after the first absorption event, when it is a distance l above the Earth, the ultimate "ruin probability," that is, the probability that it will ultimately return to the Earth rather than escape out to space is

$$P_{\mathrm{return}} = 1 - \frac{1}{N}, \tag{9}$$

where N is the total number of steps needed to escape to space. (The proof of this result is not difficult, but we skip it here for brevity; it can be found in Ref. 19.)

Perhaps surprisingly, the number N turns out to be finite, if one assumes that the density of the atmosphere decreases exponentially[20] with the height z, since in that case the attenuation length increases, also exponentially

$$l(z) = \frac{1}{\sigma n(z)} \simeq \frac{1}{\sigma n_0} e^{z/L}. \tag{10}$$

A simple hydrostatic-equilibrium treatment of the "exponential atmosphere" (such as the one found in Ref. 20) yields $n(z) = n_0 e^{-z/L}$, with a "scale height" $L = kT/mg$, where m is the molecular mass. For air, with $m \sim 29$ u, one gets, at a temperature of $T = 273$ K, a value of L of the order of 8000 m. Although CO_2 is heavier than air, it turns out to be very well-mixed, by fluid motions, throughout the atmosphere, and hence, its density may be taken to decay with the same characteristic constant as air itself. In reality, of course, the exponential decay is only a first approximation, based on an isothermal atmosphere, but we shall adopt it, for simplicity, for the remainder of this paper.[21]

Let z_j be the height of the jth "absorption layer" in our model. We have therefore

$$z_{j+1} = z_j + l(z_j) = z_j + \frac{1}{\sigma n_0} e^{z_j/L}. \tag{11}$$

This can be approximated by the differential equation $\sigma n_0 e^{-z/L} dz = dj$, which is easily integrated to show that, as $z \to \infty$, j approaches the value

$$N = \sigma n_0 L. \tag{12}$$

We can use this in Eq. (9) to conclude that

$$P_{\text{return}} \simeq 1 - \frac{1}{\sigma(\nu) n_0 L}, \tag{13}$$

where the dependence of σ on ν has been explicitly indicated. With $\sigma_0 = 3.7 \times 10^{-23}$ m^2, $n_0 = 9.91 \times 10^{21}$ molecules/m^3, and $L = 8000$ m, one gets $P_{\text{return}} = 1 - 2.7/8000 = 0.9997$, so a photon with frequency near the center of the band is virtually certain to return to Earth.

Note that under the blackbody radiation assumption, the rate of photon emission depends only on the Earth's surface temperature, not (directly) on whether the photon eventually returns or not; hence, once the photon returns, its story is effectively over. Put otherwise, one could directly multiply the Earth's emitted photon flux by $1 - P_{\text{return}}$ to get the flux out to space in this model.

The power radiated per unit area and per unit frequency by the surface of a blackbody at temperature T is given by Planck's radiation formula

$$B(\nu, T) = \frac{2\pi h \nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}, \tag{14}$$

where a factor of π has been included to account for an integral over solid angle. When this is multiplied by $1 - P_{\text{return}}$ (with the understanding that P_{return} is to be set to zero when the right-hand side of the expression (13) is negative), one obtains, for $T = 288$ K, an outgoing radiation spectrum as illustrated in Fig. 2, using the "triangular" approximation (8) for $\sigma(\nu)$.

Numerical integration of this spectrum gives the total radiation going out to space as approximately 324 W/m^2, which is 66 W/m^2, or 17%, smaller than the total intensity emitted by a 288 K blackbody (390 W/m^2). Using $x_{CO_2} = 0.17$ in Eq. (4) yields $T' = 271$ K as the temperature of a hypothetical Earth without any CO_2, under the present model, which would make CO_2 responsible for about 17 K of the total greenhouse effect. This is rather larger than the accepted values (see Sec. V for a discussion); more importantly, the actual measured spectrum of radiation coming out of the Earth differs from Fig. 2 in important ways, which indicate that the simple model we have been using so far is missing some crucial physics.

Nevertheless, this model has a number of attractive features. It provides a simple picture to illustrate how absorption, followed by reemission, does result in a net flux of photons back to Earth, at those frequencies where the atmosphere is optically thick, that is, where $N(\nu)$ is large. (Perhaps surprisingly, the existence of this "downwelling radiation" has actually been questioned by some.) It also allows one to get a first glimpse of the effect of increasing the concentration of CO_2.

Consider, for instance, a photon that, at the present CO_2 concentration, has $P_{\text{return}} = 1/2$ in Eq. (13). (According to the simplified spectrum (8), this happens around $\nu = 588$ and 752 cm^{-1}.) If one doubles n_0, P_{return} becomes 3/4, that is, the probability of the photon coming back to Earth increases by 50%, and the flux out to space at that frequency is cut in half. Clearly, this effect is important only around the wings of the CO_2 absorption spectrum: graphically, doubling the CO_2 concentration widens the range of frequencies blocked in Fig. 2 (the "hole" in the blackbody emission curve), as the inset shows (compare Fig. 4.12, top, of Ref. 3, and the description of the greenhouse gas "ditch" in Sec. 4.4.1 of the same text).

It is easy to see by direct numerical integration that the model in this section predicts a decrease in the total power radiated out to space (area under the curve in Fig. 2) of about 6.3 W/m^2 when n_0 is doubled from 390 to 780 ppm. Again this is too high; in the notation of Sec. II, this gives $\delta x_{CO_2} = 6.3/390 = 0.016$, which when used in Eq. (5) gives a climate sensitivity of about 1.9 K per doubling. The more sophisticated model in Sec. V will show how and why this estimate has to be revised downwards, but for now the present model shows that *in the absence of processes other than the ones discussed so far*, the absorptive strength of CO_2 would be sufficient to cause, by itself, an increase in temperature of the order of 2 K, if doubled from the present concentration.

V. A MORE REALISTIC MODEL: TEMPERATURE-STRATIFIED ATMOSPHERE IN LOCAL THERMODYNAMIC EQUILIBRIUM

The model presented in the previous section basically neglected the possibility of any real exchange of energy between the Earth and the atmosphere. Although formally described as absorption, the imagined scenario for a photon traveling in the atmosphere was really only a succession of elastic scattering events. (In fact, the entire "atmosphere" in that model could equivalently be replaced by a nonabsorbing mirror with a frequency-dependent transmission coefficient.)

In the real atmosphere, on the other hand, the collision time for a CO_2 molecule with an air molecule is typically shorter than the radiative transition's lifetime, which means that most of the time the photon is not reemitted immediately after it is absorbed; rather, its energy is quickly spread out among the surrounding air molecules. Of course, at a finite temperature, the reverse process is also possible: collisions with air molecules may excite the CO_2 molecules and cause them to radiate. In fact, in steady state, under the assumption of local thermodynamic equilibrium, both processes must be going on all the time, at equal rates.

The above points are made, and elaborated at some length, in the recent article by Pierrehumbert,[2] who discusses a model of layers for the atmosphere, in which each layer (at a given height and temperature) absorbs with a coefficient that depends

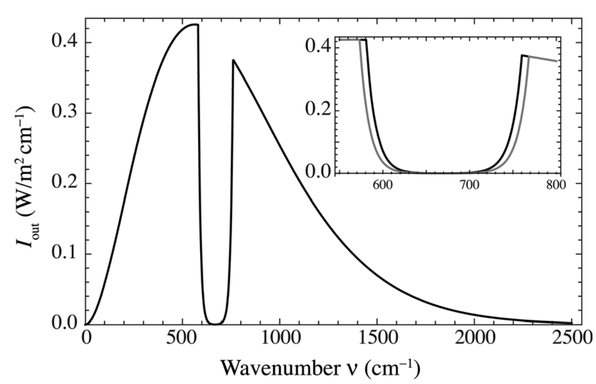

Fig. 2. The transmitted intensity through the atmosphere as a function of frequency, as given by $(1 - P_{\text{return}})B(\nu, T)$, for $T = 288$ K, using the triangular approximation (8) to the CO_2 absorption spectrum in Eq. (13). Inset: detail of the region between 550 and 800 cm^{-1} for the current CO_2 concentration of 390 ppm (black curve) and for double the concentration, or 780 ppm (gray curve).

on the frequency, and radiates (both upwards and downwards) at a rate determined by its temperature and its absorptivity. For radiation of intensity $I^{(+)}$ traveling upwards,[22] this model leads to the following differential equation:

$$\frac{dI_\nu^{(+)}}{dz} = -\alpha_\nu\left(I_\nu^{(+)} - B(\nu, T(z))\right). \tag{15}$$

Here, α_ν is the reciprocal of the absorption length l at this frequency, which, as in Sec. IV, can be written as $\alpha_\nu = n(z)\sigma(\nu) = n_0 e^{-z/L}\sigma(\nu)$. The fact that the same coefficient α_ν appears in the emission and absorption terms is an expression of Kirchhoff's law,[23] as is the presence of the blackbody radiation function $B(\nu, T)$. Equation (15), in various forms, is often referred to as Schwarzschild's equation,[8] or as the radiative transfer equation.

An important conceptual difference between this model and the one in Sec. IV is that now there is no direct connection between a photon absorbed and a photon radiated: a CO_2 molecule in the atmosphere radiates simply because it is warm, not necessarily because it has just absorbed a photon. Absorption of radiation does contribute to keep the atmosphere at a given height at a given temperature [in fact, it balances the radiative losses, as expressed in Eq. (15)], but by itself it is not enough to establish the actual temperature profile observed on Earth, as we will show later in this section. Nonetheless, many of the insights of Sec. IV do carry over to this new situation: an upwards-traveling photon that encounters many absorption layers will be unlikely to ever escape out to space, whereas a downward-traveling photon, of the same frequency, but emitted from close to the bottom of the atmosphere, will have a good chance of reaching the Earth's surface. Kirchhoff's law establishes that an object in thermal equilibrium will radiate most strongly at those frequencies where it absorbs most strongly; in the range of frequencies where absorption is very strong, therefore, the bottom of the atmosphere is like a blackbody at close to the same temperature as the surface of the Earth, radiating down to it a spectrum that is virtually identical to the one it absorbs, and hence acting (at those frequencies) essentially like a mirror, just as in Sec. IV.

To see what happens at other frequencies, and at other heights, we can formally integrate Eq. (15) as follows. As in Eq. (12), let $N(\nu) = n_0\sigma(\nu)L$ (the number of "absorption layers" we found in our earlier treatment; it may also be regarded as a sort of average "optical density of the atmosphere" at the frequency ν). Then, define the new independent variable ξ as[24]

$$\xi = \frac{1}{N(\nu)}\int_0^z \alpha_\nu(z')dz' = \left(1 - e^{-z/L}\right). \tag{16}$$

We have $\xi = 0$ at $z = 0$, and $\xi = 1$ as $z \to \infty$.

Then, $d/dz = (d\xi/dz)d/d\xi = (\alpha_\nu/N(\nu))d/d\xi$, so Eq. (15) becomes

$$\frac{dI_\nu^{(+)}}{d\xi} = -N(\nu)I_\nu^{(+)} + N(\nu)B(\nu, T(\xi)). \tag{17}$$

The formal solution of (17) is

$$I_\nu^{(+)}(\xi) = I_\nu^{(+)}(0)e^{-N(\nu)\xi} + N(\nu)\int_0^\xi e^{-N(\nu)(\xi-\xi')} \times B(\nu, T(\xi'))d\xi', \tag{18}$$

where $I_\nu^{(+)}(0)$ is the upwards radiation rate at the surface of the Earth, which, as before, we can take to be equal to $B(\nu, T(0))$ with $T(0) = 288$ K, the average surface temperature. An important approximation is that we have assumed the absorption spectrum, and hence $N(\nu)$, to be independent of height.[25] This is not exactly correct, since the broadening and the strength of the spectral lines depends on both pressure and temperature.[16] Still, for calculations based on the very rough approximation (8), this is not a significant additional shortcoming.

We are interested in estimating how much of the power emitted by the Earth's surface, $I_\nu^{(+)}(0)$, makes it out to space. The first term in Eq. (18) tells us that at the frequencies where $N(\nu)$ is large, this flux is quickly absorbed as ξ increases. As this happens, it is replaced by radiation from other atmospheric layers, at other temperatures: this is given by the second term in Eq. (18).

In order to proceed, we need to specify how the average equilibrium temperature changes with height. We shall take as guide the International Civil Aviation Organization (ICAO) "international standard atmosphere" (ISA),[26] which has a temperature lapse rate of 6.49 K/km from sea level to 11 km (the tropopause), and a constant temperature of $-56.5\,^\circ$C, or about 217 K, from 11 km up to 20 km. We can then break up the integral in (18) into three parts, one going up to 11 km, one from 11 to 20 km, where the temperature remains (approximately) constant, and one from 20 km to infinity ($\xi = 1$). The break points are at ξ_1 and ξ_2, given by

$$\xi_1 = \left(1 - e^{-z/L}\right)\Big|_{z=11000\text{m}} \simeq 0.75,$$
$$\xi_2 = \left(1 - e^{-z/L}\right)\Big|_{z=20000\text{m}} \simeq 0.92, \tag{19}$$

where the approximate values assume $L = 8000$ m.

Equation (18) then becomes, at $\xi = 1$,

$$\begin{aligned} I_\nu^{(+)}|_{\xi=1} &= B(\nu, T(0))e^{-N(\nu)} + N(\nu)\int_0^{\xi_1} e^{-N(\nu)(1-\xi')}B(\nu, T(\xi'))d\xi' \\ &\quad + \left(e^{-N(\nu)(1-\xi_2)} - e^{-N(\nu)(1-\xi_1)}\right)B(\nu, T(\xi_1)) + N(\nu)\int_{\xi_2}^1 e^{-N(\nu)(1-\xi')}B(\nu, T(\xi'))d\xi' \\ &\simeq B(\nu, T(0))e^{-N(\nu)} + N(\nu)\int_0^{\xi_1} e^{-N(\nu)(1-\xi')}B(\nu, T(\xi'))d\xi' \\ &\quad + \left(1 - e^{-N(\nu)(1-\xi_1)}\right)B(\nu, T(\xi_1)). \end{aligned} \tag{20}$$

In these equations, we have simplified the initial result by neglecting the last integral, since ξ_2 is so close to 1 that one does not expect this term to be very important except for the very largest values of $N(\nu)$. Physically, this amounts to neglecting all the contributions of the atmosphere above 20 km, on the grounds that the molecular density at those heights is very low. The main thing we lose from this approximation is a spike, at the very center of the band, due to emission by CO_2 high in the stratosphere (discussed in Pierrehumbert's article; see also Sec. VI). In keeping with this approximation, we have also set $\xi_2 = 1$ in the third term of Eq. (20), so the first exponential becomes just 1 as well; again this works except at the very center of the band.[27]

In Eq. (20), $T(\xi_1)$ is the temperature at the tropopause, that is, approximately 217 K, as stated earlier. The second integral can be numerically evaluated using the lapse rate quoted above (6.49 K/km), and the inverse relation between z and ξ: $z = -L\ln(1-\xi)$, so that

$$B(\nu, T(\xi')) = B\big(\nu, T(0) + 6.49 \times 10^{-3} L \ln(1-\xi')\big), \tag{21}$$

with L in meters.

Consider now how Eq. (20) depends on the frequency ν. The crucial parameter is the optical density $N(\nu)$. For small N, the first term dominates: the atmosphere is mostly transparent at those wavelengths, and the radiation from the Earth escapes almost unattenuated to space. For large N, on the other hand, the third term dominates, and is approximately equal to $B(\nu, T(\xi_1))$: the radiation going out to space is mostly blackbody radiation corresponding to the temperature at the tropopause.

The second term in Eq. (20) is almost never very large. When N is small, it is small because it is multiplied by N; whereas when N is large, it goes to zero because the exponential goes as $\exp[-N(\nu)(1-\xi')] \le \exp[-N(\nu)(1-\xi_1)]$, which goes to zero as N increases (since $\xi_1 < 1$). So, what the second term does is basically to interpolate between $B(\nu, T(0))$ (small N) and $B(\nu, T(\xi_1))$ (large N).

A natural interpolation scheme is

$$I_\nu^{(+)}\big|_{\xi=1} \simeq B(\nu, T(0))e^{-N(\nu)\bar{\bar{\xi}}} + \left(1 - e^{-N(\nu)\bar{\bar{\xi}}}\right) \times B(\nu, T(\xi_1)), \tag{22}$$

where $\bar{\bar{\xi}}$ is some sort of "average ξ." Two possibilities that suggest themselves are: (A) $\bar{\bar{\xi}} = (1 + (1-\xi_1))/2 = 1 - \xi_1/2$, and (B) $\bar{\bar{\xi}} = 1/[(1 + 1/(1-\xi_1))/2]$ (i.e., average the two exponentials' "decay rates," or average their reciprocals). Figure 3 shows the result of the first choice, which is extremely close to the exact result. Interestingly, Eq. (22) is essentially equivalent to Eq. (8.25) of Andrews,[5] who considers a model of a hot black surface below a "single-slab" isothermal troposphere, with a transmittance which, in our notation, would be given by $e^{-N(\nu)\bar{\bar{\xi}}}$.

Numerical integration of the interpolation (22), for the parameters indicated above and the present concentration of CO_2, yields a total flux out to space of about 339 W/m^2, which is 51 W/m^2, or about 13%, smaller than the total intensity emitted by a 288 K blackbody. Using $x_{CO_2} = 0.13$ in Eq. (4) yields $T' = 274.5$ K as the temperature of a hypothetical Earth without any CO_2, under the present model, which would make CO_2 responsible for about 13.5 K of the total (33 K) greenhouse effect. This seems still a bit on the high side, but it is closer to the right ballpark than the model in Sec. IV. For instance, Pierrehumbert's article[2] states that "for present Earth conditions, CO_2 accounts for about a third of the clear-sky greenhouse effect in the tropics and for a somewhat greater portion in the drier, colder extratropics." We note, however, that this is higher than the result recently obtained by Schmidt and co-workers[28] using a general circulation computer model, which puts the CO_2 contribution to the greenhouse effect at 24% (or about 8 K) at most.

The inset in Fig. 3 shows the effect of doubling the CO_2 concentration from present values, assuming, to lowest order, that the temperature profile of the atmosphere does not change (compare to Fig. 8.7 of Ref. 5). Clearly, as in the model in Sec. IV, and for the same reason, the range of "blocked" frequencies for radiation emitted by the surface increases, only now there is, in fact, some radiation going out to space at these frequencies. This radiation comes from the top layer of our model's atmosphere [the last term in Eq. (20)], i.e., the region between 11 and 20 km, which we have assumed to be at the same temperature as the tropopause; since this is much colder than the surface, it emits at a weaker rate. Again, numerical integration gives an area under the curve of 334 W/m^2 for the doubled CO_2 concentration, or a net decrease (keeping one more significant figure in the calculations) of about 4.2 W/m^2. This is within 15% of the current best estimate[13] of the CO_2 "radiative forcing equivalent," which, as mentioned in Sec. II, is about 3.71 W/m^2. Using $\delta x_{CO_2} = 4.2/390 = 0.011$ in Eq. (5), one gets a climate sensitivity of about 1.3 K, still a bit high, but, again, in the right ballpark.

Based on the interpolation (22) an analytical expression for the intensity "removed" from the radiation spectrum can be obtained by using a trapezoidal approximation to the area of the "hole" exhibited by the function (22). We start by identifying frequencies $\nu_\pm$ where the radiated intensity is halfway between the surface rate $B(\nu, T(0))$ and the tropopause rate $B(\nu, T(\xi_1))$, which, according to Eq. (22), happens for $e^{-N(\nu_\pm)\bar{\bar{\xi}}} = 1/2$. Using Eq. (8), we then get

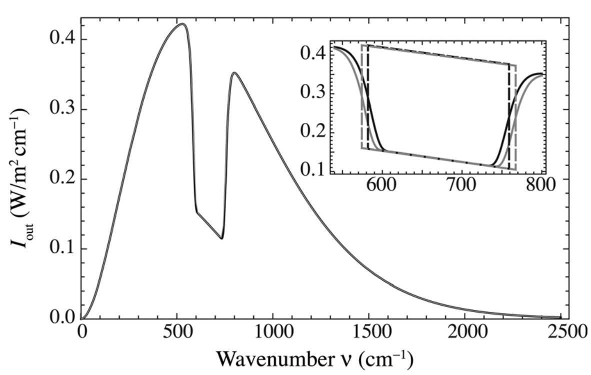

Fig. 3. The transmitted intensity through the atmosphere as a function of frequency for the model in this section, with $L = 8000$ m. Gray, thinner curve: numerical evaluation of Eq. (20). Black, thicker curve: interpolation (22), with $\bar{\bar{\xi}} = 1 + \xi_1/2$ (the two curves are virtually identical at this scale; the interpolation is very slightly wider around the bottom of the "absorption gap"). In all cases, the approximate absorption spectrum (8) has been used. Inset: detail of the gap, for the current CO_2 concentration (black curve) and for double the concentration (gray curve); the trapezoidal approximation (24) to calculate the gap area in both cases is also shown with dashed lines (black trapezoid: current concentration; gray: double the concentration).

$$\nu_{\pm} = \nu_0 \pm \frac{1}{r_{\pm}} \ln\left(\frac{n_0 \sigma_0 L}{\ln(2)/\bar{\xi}}\right). \tag{23}$$

With the parameters we have been using above, one finds $\nu_- = 582\ \mathrm{cm}^{-1}$, and $\nu_+ = 759\ \mathrm{cm}^{-1}$.

The area of the trapezoid in Fig. 4 is then

$$\begin{aligned}\Delta I \simeq & \frac{\nu_+ - \nu_-}{2}[B(\nu_+, T(0)) - B(\nu_+, T(\xi_1)) \\ & + B(\nu_-, T(0)) - B(\nu_-, T(\xi_1))] \\ \simeq & \frac{2}{r} \ln\left(\frac{n_0 \sigma_0 L}{\ln(2)/\bar{\xi}}\right)[B(\nu_0, T(0)) - B(\nu_0, T(\xi_1))],\end{aligned} \tag{24}$$

where the small difference between r_+ and r_- has been neglected, so both are replaced by their average, r, and also the average of $B(\nu_+)$ and $B(\nu_-)$ has been replaced by the value of B at the center of the band, ν_0. This very simple expression shows that the "warming power" of CO_2 increases approximately logarithmically with the concentration n_0, a well-known result that here can be seen to follow directly from the approximately exponential decay of the absorption cross section $\sigma(\nu)$ illustrated and discussed in Sec. II. Note that the result (24) is especially sensitive to the value of r, and that Fig. 1 suggests that, in the critical region of interest (around 580 and 760 cm^{-1}), the relevant slope may be steeper than the one used for the overall fit; using a larger r would reduce our estimates somewhat.

As regards the climate sensitivity, doubling the concentration n_0 in Eq. (24) yields a result that is independent of most of the model's parameters, namely

$$\delta(\Delta I) \simeq \frac{2 \ln 2}{r}[B(\nu_0, T(0)) - B(\nu_0, T(\xi_1))], \tag{25}$$

or about 4.3 W/m^2, which is consistent with the result of the numerical integration described above.

An important feature of the model considered in this section is that, in steady state, the atmosphere ends up radiating more energy (altogether, that is, adding the upwards and downwards fluxes) than it actually receives *through radiation alone* from the surface of the Earth. This is clearest in the "trapezoidal approximation," where the atmosphere is taken to be transparent for $\nu < \nu_-$ and $\nu > \nu_+$, and essentially opaque (a blackbody) for $\nu_- < \nu < \nu_+$. In this range of frequencies, therefore, the bottom of the atmosphere absorbs all the power emitted by the surface of the Earth and (since it is a blackbody at the same temperature) radiates it all back to the Earth. Power radiated at other frequencies passes through undisturbed, so it does not count. However, in the same range of frequencies, $\nu_- < \nu < \nu_+$, the top of the atmosphere clearly radiates a non-negligible additional amount of energy out to space, as a blackbody at the temperature $T(\xi_1) \simeq 217$ K (see Fig. 4); it is this radiation that is responsible for the "cooling" term in Eq. (24). The question is, Where does that extra energy come from?

The answer is that the atmospheric temperature profile that we have used in this section implicitly assumes a convective-radiative equilibrium, which includes two important warming mechanisms for the troposphere:[29] upwards convection of the air that is warmed by contact with the surface of the Earth, and condensation (which releases latent heat) of water vapor that evaporates from the surface. Both of these processes contribute to cool the Earth's surface, beyond the cooling provided by radiation alone, and hence they may be taken to be ultimately responsible for the largest part of the negative term $B(\nu_0, T(\xi_1))$ in Eqs. (24) and (25). This term, corresponding to radiation out to space from the "top of the atmosphere," is the main qualitative difference between the results in this section and the model in Sec. IV (graphically, it is the reason why the "ditch" in Fig. 3 is not nearly as deep as the one in Fig. 2). We can therefore say that the smaller climate sensitivity found in this section is due to the implicit assumption of additional, *non-radiative* cooling mechanisms (convection and evaporation) for the Earth's surface, not considered in Sec. IV.

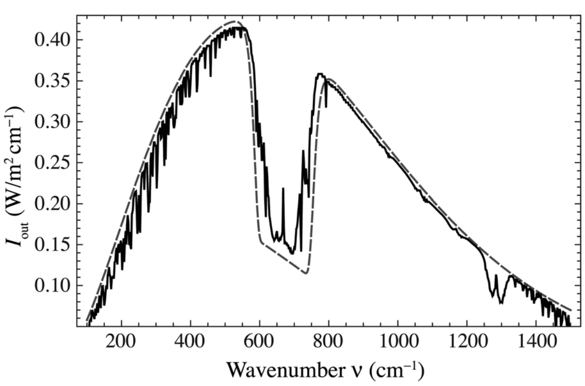

Fig. 4. MODTRAN calculation (black, solid curve), and our approximation (dashed, gray curve); see text for details.

VI. COMPARISON WITH MORE ACCURATE CALCULATIONS

All our results are based on a very simplified treatment of a particular spectral region of the CO_2 absorption spectrum. Naturally, the real world is more complicated in many ways. Not only is the CO_2 spectrum, as already shown in Sec. II, much more complex than the simple approximation (8), but it also changes somewhat with atmospheric pressure and contains additional absorption lines in other spectral regions. There are also other greenhouse gases in the atmosphere that contribute absorption bands of their own, and at least in one important case (water vapor below 600 cm^{-1}) overlap with the CO_2 lines to a non-negligible extent. Finally, the spatial non-uniformity of the temperature profile of the Earth's atmosphere cannot be neglected in any serious model: important properties such as the Earth's surface temperature and the height of the tropopause vary considerably around the world at any given time.

Given all this, the relatively good agreement between the results of our model and more sophisticated ones has to be regarded, at least in part, as fortuitous. To some extent, however, we may expect the errors introduced by some of our approximations to cancel each other. For instance, we have used a purely one-dimensional model, whereas in real life radiation may travel through the atmosphere in all directions. Generally speaking, this tends to increase the effective optical density of the atmosphere (since an oblique ray travels through a greater thickness of air before leaving the atmosphere). On the other hand, we have also neglected the temperature and pressure dependence of the molecular CO_2 absorption spectrum, using only a result valid at standard pressure and temperature; higher in the atmosphere, at lower pressures and temperatures, the lines tend to narrow and their absorbing strength decreases. This error, therefore, would at least in part tend to cancel out the previous one.

Since it is not our intention to turn this into a research paper, we have deliberately avoided all these complications, but it would be natural for the reader to wonder how our approximations actually compare to the results of more sophisticated calculations. David Archer, of the University of Chicago, has set up a Web interface[9] to a "narrow band model" atmospheric radiative transfer code called MODTRAN (developed by Spectral Sciences and the U.S. Air Force), which solves the radiative transfer equations for a variety of possible scenarios. We have found it instructive to "play" with this simulator and compare its results to ours. In this section, we present some of the highlights of such comparisons; the reader is encouraged to experiment further on his or her own.

The simulator has a few preset scenarios. The closest to ours is the one labeled "1976 US Standard Atmosphere," and for the best agreement with our calculations, one should set the concentrations of all the other greenhouse gases, such as CH_4 and ozone, equal to zero. To completely remove water vapor, set "Water Vapor Scale" to zero as well. Also set the CO_2 concentration to 390 ppm. The result of the MODTRAN calculation is then shown in Fig. 4, along with our approximation (22) (dashed line).

Clearly, the qualitative agreement is quite good, although there are important discrepancies as well. The bottom of the absorption gap, as calculated by MODTRAN, seems to correspond to emission at a higher temperature (about 220 K) than the 217 K we have used for our approximations. The sharp spike in the middle of the absorption band, mentioned earlier, and discussed in Pierrehumbert's article, is due to emission by CO_2 in the (much warmer) upper stratosphere; interestingly, it can be made to vanish from the MODTRAN calculation by setting the "sensor altitude" low enough, for instance, at 20 km; this also seems to bring the bottom of the gap generally a little lower down. The "real" gap also appears to be somewhat narrower than in our calculations, and with rougher sides. There is also a secondary CO_2 absorption feature around 1284 cm^{-1} (7.79 μm).

The MODTRAN calculator also produces a result for I_{out}, the total flux out to space, for the spectral interval shown (100 to 1500 cm^{-1}), which in this case is about 312.9 W/m^2. Doubling the CO_2 concentration reduces I_{out} to 309.4 W/m^2, so according to this calculation $\delta(\Delta I) = 3.5$ W/m^2, somewhat lower than the "canonical" CO_2 radiative forcing equivalent of 3.71 W/m^2. The discrepancy is even more noticeable when MODTRAN is run with other greenhouse gases reset to their default values: 1.7 ppm of CH_4, 28 ppb of tropospheric ozone, stratospheric ozone scale $= 1$ and water vapor scale $= 1$. With these settings, and 390 ppm of CO_2, one gets for the "1976 US Standard Atmosphere" scenario $I_{out} = 258.7$ W/m^2, which goes down to 255.8 W/m^2 for 780 ppm of CO_2, for a change of less than 3 W/m^2. This would suggest a $\delta x_{CO_2} \simeq 3/390 \simeq 0.008$, which substituted in Eq. (5) gives a no-feedback climate sensitivity of about 0.9 K. Note, however, that other scenarios (such as "tropical atmosphere") yield still different results, showing that disentangling the effect of CO_2 alone from all the other "real world" forcings is not a trivial task.

VII. CONCLUSIONS

We have shown here how a couple of simple physical models, together with basic observational data, can be used to establish the importance of CO_2 as a greenhouse gas in the Earth's present-day atmosphere. The input data that we have used are the absorption spectrum of molecular CO_2, and, in Sec. V, the temperature profile of the "standard atmosphere." We have pointed out that the latter cannot be explained, as far as energy conservation is concerned, by radiative physics alone, and therefore would require, if one tried to derive it from first principles, a consideration of other heat transfer mechanisms, such as convection and evaporation; nonetheless, taking this "lapse rate" as a given, we have shown how radiative physics then leads in a fairly straightforward manner to an estimate for the CO_2 radiative forcing that is not very different from the currently accepted best value. We have also, in Sec. IV, considered a model without any convective cooling, whose results may be regarded as providing an upper bound to the true "no-feedback" climate sensitivity.

The question of feedbacks, in its broadest sense, is the whole question of climate change: namely, how much and in which way can we expect the Earth to respond to an increase of the average surface temperature of the order of 1 degree, arising from an eventual doubling of the concentration of CO_2 in the atmosphere? And what further changes in temperature may result from this response? These are, of course, questions for climate scientists to resolve. We can only say, from the results presented here, that such a doubling would be initially (that is, before any feedbacks "kick in") equivalent to an increase of about 4 W/m^2 in the average solar irradiance at the Earth's surface. This is like increasing the quantity I_0 on the right-hand-side of Eq. (2) by about 22 W/m^2; or, equivalently, increasing the sun's brightness by about 1.6%.

ACKNOWLEDGMENTS

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[2]R. T. Pierrehumbert, "Infrared radiation and planetary temperature," Phys. Today **64**, 33–38 (2011).

[3]R. T. Pierrehumbert, *Principles of Planetary Climate* (Cambridge University, Cambridge, England, 2010).

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[7]Relegating most of them to footnotes, like this one.

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[10]See also, for instance, Sec. 1.3.2 of Ref. 5.
[11]A. P. Smith, "Proof of the atmospheric Greenhouse effect," e-print arXiv:0802.4324v1 [physics.ao-ph].
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[21]A slightly more sophisticated model for the atmospheric density, which included the effect of the temperature lapse rate, was used in the numerical calculations in Ref. 6.
[22]We are still considering only a one-dimensional model, so "upwards" here means "precisely vertically." This, and other limitations of our model, will be revisited at the beginning of the next section.
[23]F. Reif, *Fundamentals of Statistical and Thermal Physics*, Sec. 9.15 (McGraw-Hill, New York, 1965).
[24]The product $N(\nu)\xi(z)$ is related to the "optical depth" $\chi_\nu(z)$ at wavenumber ν, measured downward from the top of the atmosphere (in Andrews 5, Chap. 3), by $N(\nu)\xi(z) = \chi_\nu(0) - \chi_\nu(z)$. It is similarly related to Pierrehumbert's τ_ν, which increases with altitude (Ref. 3, Sec. 4.2) as $\tau_\nu(p(z), p(0))$.
[25]In fact, Eq. (18) is equivalent to the first Eq. (4.9) of Ref. 3, if the dependence of $N(\nu)$ on height is neglected there. The reader is referred to Chap. 4 of Ref. 3 for a much more in-depth treatment of the radiative transfer equations, with many examples.
[26]International Civil Aviation Organization, *Manual of the ICAO Standard Atmosphere (extended to 80 kilometres (262 500 feet))*, Doc 7488-CD, Third Edition, 1993, ISBN 92-9194-004-6.
[27]We note that the climate literature, by convention, actually considers the radiative forcing at the tropopause, rather than at $z = \infty$, so in fact $I_\nu^{(+)}|_{\xi=\xi_1} = B(\nu, T(0))e^{-N(\nu)\xi_1} + N(\nu)\int_0^{\xi_1} e^{-N(\nu)(\xi_1-\xi')}B(\nu, T(\xi'))d\xi'$ would be a better quantity to compare with standard climate models. Interestingly, although this obviously has the same large and small $N(\nu)$ limits as Eq. (20), we have not been able to find a simple approximation to it that works as well as Eq. (22) does for Eq. (20).
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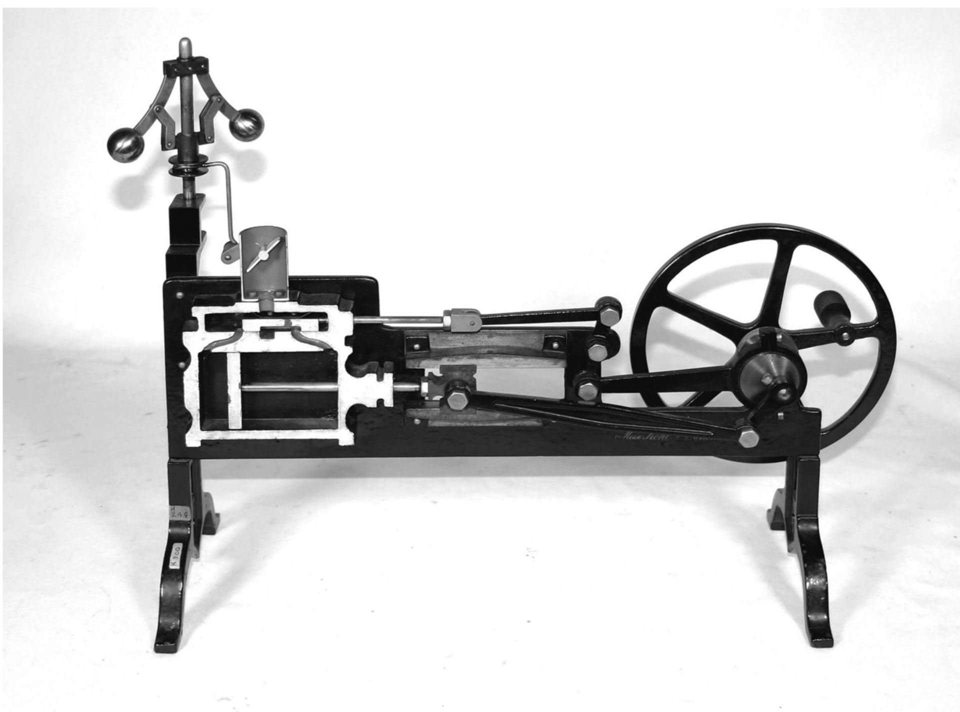

Steam Engine Half Model. This model is described in the 1900 catalogue of Max Kohl of Chemnitz, Germany as a "sectional model of a horizontal steam engine with distributing-valve regulator and throttle-valve, with handle for turning" and cost 54 Marks, the equivalent of about $14. The important item is the governor, which is linked to the tilting valve in the steam intake at the top of the cylinder mechanism; this is an early example of a feed-back mechanism. The model came to the Greenslade Collection from Wellesley College. (Notes and photograph by Thomas B. Greenslade, Jr., Kenyon College)